(JAPLA 2008/06/28)

On the problem of "PARTITION of INTEGERS

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Here is 4 pieces of ORANGE. Suppose that they take these sharing by somebody. How many methods of sharing can be considered ?

i) 4 is taken by one person

ii) 3 is taken by one person and 1 is taken by another person

iii) 2 are taken respectively by two persons

iv) 2 is taken by one person and 1 are taken by other two persons

v) 4 are taken separately by four persons

Thus there are 5 methods. We write this answer as follows :

4	3 1	2 2	2 1 1	1 1 1 1

This is the problem of " PARTITION of INTEGERS ".

The function "part" is defined as follows::

```
   dev=:3 :0

r=.y;<"1(y-t),.t=.>:i.y-h=.<.-:>:ywhile.h>1
   do.r=.r,<(s>0)#s=.s,y-+/s=.(<.y%h)$h=.h-1
end.
)
```

```
   next=;3 0

m=.>:#(1=s)#s=.>y
h=.((#s)-m){.s [ t=.(-m){.s=.>y
<h,+/&>(-<:{.t)<\(+/t)$1
)
```

```
   devide=:3 :0

t=.r=.y
while.(+/}.s)>#(s=1)#s=.>t
   do.r=.r,t=.next t
end.
)
```

```
   part=:3 :0

r=.2{.s [ t=.(-(#s)+<:k=._1){.s=.dev y
while.k<y-3
   do.r=.r,devide(k=.k+1){ t
end.
)
```

We can confirm that "part" is well defined by the following examples.

```
   part 4
```

4	3 1	2 2	2 1 1	1 1 1 1

part 5

5	4 2	3 2	3 1 1	2 2 1	2 1 1 1	1 1 1 1 1

10 4 $ p10=:part 10

10	9 1	8 2	8 1 1
7 3	7 2 1	7 1 1 1	6 4
6 3 1	6 2 2	6 2 1 1	6 1 1 1 1
5 5	5 4 1	5 3 2	5 3 1 1
5 2 2 1	5 2 1 1 1	5 1 1 1 1 1	4 4 2
4 4 1 1	4 3 3	4 3 2 1	4 3 1 1 1
4 2 2 2	4 2 2 1 1	4 2 1 1 1 1	4 1 1 1 1 1 1
3 3 3 1	3 3 2 2	3 3 2 1 1	3 3 1 1 1 1
3 2 2 2 1	3 2 2 1 1 1	3 2 1 1 1 1 1	3 1 1 1 1 1 1 1
2 2 2 2 2	2 2 2 2 1 1	2 2 2 1 1 1 1	2 2 1 1 1 1 1 1

_2 {. p10

2 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1

$ p11=:part 11

56

_2 {. p11

2 1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1 1

$ p12=:part 12

77

_2 {. p12

2 1 1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1 1 1

【computing times of the function "part"】

We can not confirm that the new function of "M." version J6.02 fulfill exactly. NB. tc=:6!:2		
tc'P10=:part 10' 0.0101714 \# P10 42	tc'P20=:part 20' 0.110142 \# P20 627	tc'P30=:part 30' 1.41232 \# P30 5604
tc'P40=:part 40' 32.2255 \# P40	tc'P50=:part 50' 1719.47 \# P50	Computation time increases abruptly as n increases.

37338	204226	

When we change ”part” to the following “partno” , computation time is shortened.

devno=:3 :0
 	
r=.1 [t=.y
while.(+/}.s)>#(s=1)#s=.>t
do.r=.r+#t=.next t
end.
)
	partno=:3 :0
	
	t=.(-(#s)-(r=.3)+k=._1){.s=.dev y
	while.k<y-4
	do.r=.r+devno(k=.k+1){t
	end.
	)

tc'n40=:partno 40' 4.80605 n40 37338	tc'n60=:partno 60' 123.875 n60 966467	tc'n80=:partno 80' 2567.82 n80 15796476
tc'n100=:partno 100' 31109 n100 190569292	24 60 60 #: 31109 8 38 29 When n=100, computation time of “partno” takes about 9 hours !	

We note that there are some pair of integers where the recurrence relation does not hold.

For exanple,

102-L:0 no 102

1 2	5 7	12 15	22 26	35 40	51 57	70 77	92 100	‘ ‘

$p(5) = \_1 + p(4) + p(3) = \_1 + 5 + 3 = 7$

$p(7) = \_1 + p(6) + p(5) - p(2) = \_1 + 11 + 7 - 2 = \_1 + 16 = 15$

$p(12) = 1 + p(11) + p(10) - \{p(7) + p(5)\} = (56 + 42) - (15 + 7) = 1 + 76 = 77$

$p(15) = 1 + \{p(14) + p(13)\} - \{p(10) + p(8)\} + p(3) = 135 + 101 - (42 + 22) + 3 = 176$

$$p(22) = \_1 + \{p(21) + p(20)\} - \{p(17) + p(15)\} + \{p(10) + p(7)\}$$
$$= \_1 + \{792 + 627\} - \{297 + 176\} + \{42 + 15\} = \_1 + \{1419\} - \{473\} + \{57\} = 1002$$

$$p(26) = \_1 + \{p(25) + p(24)\} - \{p(21) + p(19)\} + \{p(14) + p(11)\} - p(4)$$
$$= \_1 + \{1958 + 1575\} - \{792 + 490\} + \{135 + 56\} - 5 = \_1 + 3533 - 1282 + 191 - 5 = 2436$$

$$p(35) = 1 + \{p(34) + p(33)\} - \{p(30) + p(28)\} + \{p(23) + p(25)\} - \{p(13) + p(19)\}$$
$$= 1 + \{12310 + 10143\} - \{5604 + 3718\} + \{1255 + 627\} - \{101 + 30\} = 14883$$
$$p(40) = 1 + \{p(39) + p(38)\} - \{p(35) + p(33)\} + \{p(28) + p(25)\} - \{p(18) + p(14)\} + p(5)$$
$$= 1 + \{31185 + 26015\} - \{14883 + 10143\} + \{3718 + 1958\} - \{385 + 135\} + 7 = 37338$$

We find that the best computational recursive method as follows :

r10	]pp=:(<:s){r10	]qq=:(<: t){r10	p-q
1 2 3 5 7 11 15 22 30 42]'s t'=:red 11	42 30]p=:+/pp 72	11 5]q=:+/qq 16	56 this "p11"
s: 10 9 / t: 6 4			

{:r11=:r10,p-q	]pp=:(<:s){r11	]qq=:(<:t){r11	p-q
56 's t'=:red 12	56 42]p=:+/pp 98	15 7]q=:+/qq 22	76 1+p-q 77 this "p12"
s: 11 10 / t: 7 5			
Note that "r12" is " 1+p-q" added to "r11"			

{:r12=:r11,1+p-q 77 's t'=:red 13	]p=:+/(<:s){r12 134	]q=:+/(<:t){r12 33	p-q 101 "p13"
{:r13=:r12,p-q 101 's t'=:red 14	]p=:+/(<:s){r13 180	]q=:+/(<:t){r13 45	p-q 135 "p14"
{:r14=:r13,p-q 135 's t'=:red 15	]p=:+/(<:s){r14 239	]q=:+/(<:t){r14 64	1+p-q 176 "p15"
{:r15=:r14,1+p-q 176 's t'=:red 16	]p=:+/(<:s){r15 317	]q=:+/(<:t){r15 86	p-q 231 "p16"

When we change " partno" to the following "part_no" , the computation time is just drastically shortened.

```
  no=:3 :0
r=.<1,1+s=.(k=.0){t=.+:&.>:i.y
while.y>s
  do.r=.r,<s,(+>:k)+s=.((k=.k+1){t)+{:>{:r
end.
((>0:)#])L:0 y-L:0}:r
)
```

```
   pq=:3 :0
r=.<12,12+s=.(k=.0){t=.+:&.>:1+i.b=.>.y%10
while.y>{.>{:r
  do.r=.r,<s,(3+k)+s=.2+((k=.k+1){t)+{:>{:r
end.
qq=.r-.pp=.(+:i.>.-:#r){r
((p<:y)#p=.;pp);(q<:y)#q=.;qq
)
```

```
   part_no=:3 :0
k=.#r=.1 2 3 5 7 11 15 22 30 42
if.y<11 do.(<:y){r else.'p q'=.pq y
   while.k<y
         do.s=.(+/(<:>{.h){r)-+/(<:>{:h=.red k=.k+1){r
         {:r=.r,s+(k e.p)-k e.q
   end.
end.
)
```

```
   red=:3 :0
if.y<13 do. r=.no y else.
   t=.((>.k%2),2)$c=.i.k=.#r=.no y
   p=.;(;{."1 t){r
   p;;((s>0)#s=.{:"1 t){r
end.
)
```

```
r30=:part_no"0>:i.30

   no 26
 | 25 24 | 21 19 | 14 11 | 4 |        ]'s
t'=:red 26
 | 25 24 14 11 | 21 19 4 |
pp;p=:+/pp=:(<:s){r30
 | 1958 1575 135 56 | 3724 |
qq;q=:+/qq=:(<:t){r30
 | 792 490 5 | 1287 |       p - q
2437
   ]'a b'=:pq 26
 | 12 15 | 22 26 |      (p-q)+(26 e.a)-26 e.b
2436
   25 { r30
2436
just " p26"  for n=26 !
```

3 10 $ part_no″0 >:i.30 1 2 3 5 7 11 15 22 30 42 56 77 101 135 176 231 297 385 490 627 792 1002 1255 1575 1958 2436 3010 3718 4565 5604	$p(1) \sim p(30)$
part_no″0(10*4+>:i.7) 204226 966467 4087968 15796476 56634173 190569292 607163746	p40～p100(10)

tc'p200=:x:part_no 200' 0.193761 p200 3972999029388 tc'p300=:x:part_no 300' 0.665288 p300 9253082936723528 tc'p400=:x:part_no 400' 0.554037 p400 6727090051588284416 tc'p500=:x:part_no 500' 0.803441 p500 2300165087559805829120 tc'p600=:x:part_no 600' 1.39866 p600 458005063749494810607616 tc'p700=:x:part_no 700' 1.41037 p700 60378645638998065139417088	tc' p800=:x:part_no 800' 1.76496 p800 5733259907364596643528704000 tc'p900=:x:part_no 900' 2.1649 p900 415935058092506687910289866752 tc'p1000=:x:part_no 1000' 2.61234 p1000 24070293522200282323346880724992 tc'p1500=:x:part_no 1500' 5.73519]P1500=:}." :p1500 892120554652152127515929766651194507264 tc'p2000=:x:part_no 2000' 9.46603 (Note that computation time is less than ten second !)]p2000=:}." :p2000 4242090736196891424949948886323686243393769308160 $&> p1500;p2000 39 49